

# On the Branching Structure of Logical Systems Generated by Time-Evolving Axiom Systems

## 1 Introduction

In this paper, we consider “time-evolving axiom systems,” in which axioms are added over time, and study the branching structure of the logical systems they generate.

In particular, we establish the following:

- There exist infinitely many finite and infinite axiom systems
- Even systems sharing the same axioms can differ infinitely
- The order of axiom acquisition affects intermediate logical results
- The future of logical systems branches infinitely under temporal evolution

## 2 Basic Definitions

**Definition 1.** An axiom system over a formal language  $L$  is a pair  $(\Gamma, R)$ , where  $\Gamma$  is a set of sentences and  $R$  is a set of inference rules.

**Definition 2.** For a set of axioms  $\Gamma$ , the set of all theorems derivable from  $\Gamma$  is denoted by

$$\text{Th}(\Gamma).$$

**Definition 3.** A time-evolving axiom system is a family of axiom sets  $\{A_t\}_t$  such that

$$s \leq t \Rightarrow A_s \subseteq A_t.$$

### 3 Infinitude of Finite Axiom Systems

**Theorem 1.** There exist infinitely many finite axiom systems.

*Proof.* Take infinitely many distinct sentences  $\varphi_0, \varphi_1, \varphi_2, \dots$ . For each  $n$ , define  $\Gamma_n = \{\varphi_n\}$ . Each  $\Gamma_n$  is finite, and  $\Gamma_n \neq \Gamma_m$  for  $n \neq m$ . Thus, there are infinitely many finite axiom systems.  $\square$

### 4 Branching of Infinite Axiom Systems

**Theorem 2.** There exist infinitely many infinite axiom systems sharing a common finite subset  $\Gamma_0$ .

*Proof.* Let  $\beta_0, \beta_1, \beta_2, \dots$  be infinitely many distinct sentences not in  $\Gamma_0$ . Define

$$\Gamma_n = \Gamma_0 \cup \{\beta_m \mid m \geq n\}.$$

Each  $\Gamma_n$  is infinite, and  $\Gamma_n \neq \Gamma_m$  for  $n \neq m$ . Thus, infinitely many such systems exist.  $\square$

### 5 Branching of Future Logical Systems

**Theorem 3.** If infinitely many axioms can be added after some time  $t_0$ , then the future of the logical systems generated by a time-evolving axiom system is infinite.

*Proof.* Let  $G$  be the common axiom set at time  $t_0$ , and let  $\beta_0, \beta_1, \beta_2, \dots$  be distinct sentences.

For each  $n$ , define

$$A_t^{(n)} = \begin{cases} G & (t \leq t_0) \\ G \cup \{\beta_m \mid m \geq n, \tau_m \leq t\} & (t > t_0) \end{cases}$$

If  $n < m$ , then  $\beta_n$  appears in system  $n$  but not in system  $m$ , hence their theorem sets differ.

Thus, infinitely many distinct futures exist.  $\square$

### 6 Dependence on Order of Axiom Acquisition

**Theorem 4.** Different orders of acquiring axioms yield different logical results at some time.

*Proof.* Let  $G$  be a common initial axiom set, and let  $\alpha, \beta$  be independent over  $G$ :

$$G \cup \{\alpha\} \not\vdash \beta, \quad G \cup \{\beta\} \not\vdash \alpha.$$

Consider two systems:

- System 1: add  $\alpha$  first
- System 2: add  $\beta$  first

At an intermediate time:

$$A^{(1)} = G \cup \{\alpha\}, \quad A^{(2)} = G \cup \{\beta\}.$$

In System 1,  $\alpha$  is provable; in System 2, it is not. Thus,

$$\text{Th}(A^{(1)}) \neq \text{Th}(A^{(2)}).$$

□

## 7 Future Forms of Thought

**Theorem 5.** If human thought depends on multiple formal systems and admits infinitely many possible extensions, then its future forms are infinite.

*Proof.* Let  $\delta_0, \delta_1, \delta_2, \dots$  be extension candidates. For each  $n$ , define a system adopting  $\delta_n, \delta_{n+1}, \dots$

Since different  $n$  lead to different extensions, the resulting forms differ.

□

## 8 Conclusion

In time-evolving axiom systems, logical systems are not uniquely determined but instead branch infinitely depending on:

- choice of axioms,
- order of acquisition,
- selection of formal systems.

This reveals an intrinsic multiplicity of dynamic logical systems, distinct from the limitations imposed by Gödel's incompleteness in fixed systems.